The particle kinetic energy varies as

$$\frac{\partial}{\partial t}\frac{1}{2}\rho v^2 = \delta \mathbf{v}\cdot\mathbf{E} = \mathbf{j}\cdot\mathbf{E}$$

which can be written

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{\mathrm{B}^2}{8\pi}\right) + \nabla\cdot\mathbf{P} = 0$$

with the electromagnetic energy flux given by $\mathbf{P}$.

Note that, if initially each particle is represented by a localized spike in ρ and δ, it is an easy matter to average over a local volume V because the equations are all linear in ρ and δ.

Note that if $\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c$

then $\mathbf{P} = \mathbf{v}_{\perp}\frac{\mathrm{B}^2}{4\pi}$

Where $\mathbf{v}_{\perp}$ is the velocity $\perp \mathbf{B}$

Note that **E** plays no significant dynamical role.

$$\frac{E^2}{8\pi} = O\left(\frac{v^2}{c^2}\right)\frac{B^2}{8\pi}$$

Note, too, that the existence of **E** depends upon what frame of reference the calculation uses.

See example in V. Vasyliunas, 2001, Geophys. Res. Letters, **28**, 2177.

Similarly **j** plays no dynamical role because it has no energy and no strength.

Note that in any real gaseous medium **j** is driven by a weak **E**', pulling energy out of the magnetic field.

B causes **j**, not vice versa.

Note that **j** is driven by the relation

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi \mathbf{j}$$

Note, too, that $\frac{E'}{B} \sim \frac{c}{4\pi \ell \sigma} \sim \frac{10^{-4}}{\ell}$ at 10^4 K.

To estimate E'/B in specific cases, note that

$$\sigma \approx 2\times10^{7}T^{3/2}\sec^{-1}.$$

So in a magnetic field B with characteristic scale l we have

$$\frac{E'}{B} \approx \frac{10^{-4}}{l}\left(\frac{10^{4}}{T}\right)^{3/2}.$$

For T not less than 10^4 K and l as small as 100 km, it follows that

$$\frac{E'}{B} \leq 10^{-11}.$$

The stress and energy in the electric field is no more than 10^{-22} times the energy and stress in the magnetic field.

Note that the dynamics is all in terms of

$$\rho = \mathrm{NM}, \quad \mathbf{p} = \mathrm{NkT}, \quad \mathbf{v}, \quad \text{and } \mathbf{B}$$

when $\mathbf{E} = -\mathbf{v} \times \mathbf{B} / c$.

$\mathbf{E}$ and $\mathbf{j}$ are secondary passive quantities.

Consider the role of the neglected $\mathbf{E}'$.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - c\nabla \times \mathbf{E}'$$

For a collision dominated plasma, $\mathbf{j} = \sigma \mathbf{E}'$.

Hence $\mathbf{E}' = \dfrac{c}{4\pi\sigma} \nabla \times \mathbf{B}$ and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

$$\eta = \frac{c^2}{4\pi\sigma} \sim 0.5 \times 10^{13} / \mathrm{T}^{\frac{3}{2}}\ \mathrm{cm}^3/\mathrm{sec}$$

Magnetic Reynolds number

$$\mathrm{N_R} = \frac{\mathrm{vL}}{\eta}$$

For large $\mathrm{N_R}$ the principal effect is bulk transport of **B.**

For a partially ionized gas

N = number density of neutral atoms,
n = number density of ions/electrons

v = mean bulk velocity of neutral gas

w = mean bulk velocity of ions

u = mean bulk velocity of electrons

τ_i = ion-neutral collision time

τ_e = electron-neutral collision time

τ = ion-electron collision time

p = pressure of neutral gas

$$NM\frac{dv}{dt} = -\nabla p + \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_c}$$

Consider a slightly ionized gas, $n<<N$.

Neglect ion and electron pressures.

$$m\frac{d\mathbf{u}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{u}\times\mathbf{B}}{\mathbf{c}}\right) - \frac{\mathrm{m}(\mathbf{u}-\mathbf{v})}{\tau_c} - \frac{\mathrm{m}(\mathbf{u}-\mathbf{w})}{\tau}$$

$$\mathrm{M}\frac{d\mathbf{w}}{dt} = +e\left(\mathbf{E} + \frac{\mathbf{w}\times\mathbf{B}}{\mathbf{c}}\right) - \frac{\mathrm{m}(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{\mathrm{m}(\mathbf{u}-\mathbf{w})}{\tau}$$

$$\mathbf{j}=\mathrm{ne}(\mathbf{w}-\mathbf{u})$$

From Ampere's law

$$\mathbf{u} = \mathbf{w} - \frac{\mathbf{c}}{4\pi ne}\nabla\times\mathbf{B}$$

Neglect the electron and ion inertia. The sum of the two eqns. of motion gives

$$\frac{\mathrm{nM}(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{\mathrm{nm}(\mathbf{u}-\mathbf{v})}{\tau_e} = \mathrm{en}(\mathbf{w}-\mathbf{u})\times\frac{\mathbf{B}}{c}$$

$$= \mathbf{j}\times\mathbf{B}/c = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mathbf{4}\pi}$$

Hence, for the neutral atoms

$$\mathrm{NM}\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{dt}} = -\nabla p + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}$$

which is the usual MHD momentum eqn.

Note that

$$\mathbf{w} = \mathbf{v} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi n \mathrm{Q}} + \frac{c\,\mathrm{m}/\tau_e}{4\pi n e \mathrm{Q}} \nabla \times \mathbf{B}$$

$$\mathbf{u} = \mathbf{v} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi n \mathrm{Q}} - \frac{c\,\mathrm{M}/\tau_i}{4\pi n e \mathrm{Q}} \nabla \times \mathbf{B}$$

where

$$\mathrm{Q} \equiv \frac{\mathrm{M}}{\tau_i} + \frac{\mathrm{m}}{\tau_e} \cong \frac{\mathrm{M}}{\tau_i}$$

Then

$$\mathbf{E} = -\frac{\mathbf{v} \times \mathbf{B}}{\mathbf{c}} - \frac{[(\nabla \times \mathbf{B}) \times \mathbf{B}] \times \mathbf{B}}{4\pi n c \mathrm{Q}} + \frac{\mathrm{M}/\tau_i - m/\tau_e}{4\pi n e \mathrm{Q}} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$+ \frac{c}{4\pi n e^2} \left[\frac{(\mathrm{M}/\tau_i)(\mathrm{m}/\tau_e)}{\mathrm{Q}} + \frac{m}{\tau} \right] \nabla \times \mathbf{B}$$

Define

$$\alpha \equiv \frac{c\mathrm{B}}{4\pi n e} \left[\frac{\mathrm{M}/\tau_i - \mathrm{m}/\tau_e}{\mathrm{M}/\tau_i + \mathrm{m}/\tau_e} \right]$$ Hall coefficient

$$\beta \equiv \frac{\mathrm{B}^2}{4\pi n \mathrm{Q}}$$ Pedersen coefficient

$$\eta \equiv \frac{c^2}{4\pi n e^2} \left[\frac{(\mathrm{M}/\tau_i)(\mathrm{m}/\tau_e)}{\mathrm{M}/\tau_i + \mathrm{m}/\tau_e} + \frac{\mathrm{m}}{\tau} \right]$$ Ohm's coefficient

Write **b** = **B**/B, so that

$$\mathbf{E} = \frac{B}{c}\left[-\mathbf{v}\times\mathbf{b} - \beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} + \eta\nabla\times\mathbf{b}\right]$$

$$\mathbf{E}' = \frac{B}{c}\left[\eta\nabla\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} - \beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b}\right]$$

The induction equation

$$\partial\mathbf{B}/\partial t = -c\nabla\times\mathbf{E}$$

becomes

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b}) + \nabla\times\{\beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b} - \alpha(\nabla\times\mathbf{b})\times\mathbf{b}\}$$

This is the usual MHD eqn. with two extra terms.

In terms of the non-dimensional Lorentz force

$$\mathbf{f} = \frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{4\pi}$$

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b}) + \nabla\times[\beta\mathbf{f}\times\mathbf{b} - \alpha\mathbf{f}]$$

The magnetic energy equation can be written

$$\frac{\partial}{\partial t}\left(\frac{1}{8\pi}\mathbf{b}^2\right)+\nabla\cdot\left[\frac{v_\perp \mathbf{b}^2}{4\pi}+\eta\mathbf{f}+\beta\mathbf{b}^2\mathbf{f}+\alpha\mathbf{f}\times\mathbf{b}\right]$$

$$=-\mathbf{v}\cdot\mathbf{f}-\frac{\eta(\nabla\times\mathbf{b})^2}{4\pi}-4\pi\beta\mathbf{f}^2$$

The right hand side represents the dissipation of magnetic energy. The term in square brackets represents the transport of magnetic energy.

Consider the Hall effect with $\beta=\eta=\nabla\alpha=0$

$$\frac{\partial \mathbf{b}}{\partial \mathrm{t}}=\nabla\times(\mathbf{v}\times\mathbf{b})-4\pi\alpha\nabla\times\mathbf{f}$$

$$\frac{\partial \mathbf{v}}{\partial \mathrm{t}}+\boldsymbol{\omega}\times\mathbf{v}=-\frac{\nabla \mathrm{p}}{\mathrm{NM}}-\nabla\left(\tfrac{1}{2}v^2\right)+4\pi\,\mathrm{C}^2\,\mathbf{f}$$

$$\mathrm{C}^2=\mathrm{B}^2/(4\pi\,\mathrm{NM}),\quad \boldsymbol{\omega}=\nabla\times\mathbf{v}$$

Then
$$\frac{\partial \boldsymbol{\omega}}{\partial t}=\nabla\times(\mathbf{v}\times\boldsymbol{\omega})+4\pi\,\mathrm{C}^2\,\nabla\times\mathbf{f}$$

Hence

$$\frac{\partial}{\partial \mathrm{t}}\left(\mathbf{b}+\frac{\alpha}{\mathrm{C}^2}\boldsymbol{\omega}\right)=\nabla\times\left[\mathbf{v}\times\left(\mathbf{b}+\frac{\alpha}{\mathrm{C}^2}\boldsymbol{\omega}\right)\right]$$

Note that the Hall (vorticity) contribution is smaller O(1/L) compared to the magnetic field. And that makes it the same order as resistive diffusion.

$$\frac{\alpha}{\eta} \sim \Omega_e \tau_e \qquad\qquad \Omega_e = \frac{eB}{mc}$$

The Hall effect is a small-scale effect.

See JGR, **101,** 10587-10625, (1996).

If the ion and electron pressures, inertia, and other applied forces $\mathbf{L}_i$, $\mathbf{L}_e$ per unit mass are included, then

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) - \nabla \times [\eta \nabla \times \mathbf{b} - \beta \mathbf{f} \times \mathbf{b} + \alpha \mathbf{f}]$$

$$- \frac{cm/\tau_e}{eBQ} \nabla \times \left[\frac{\nabla p_i}{n} + M\left(\frac{d\mathbf{w}}{dt} - \mathbf{L}_i \right) \right]$$

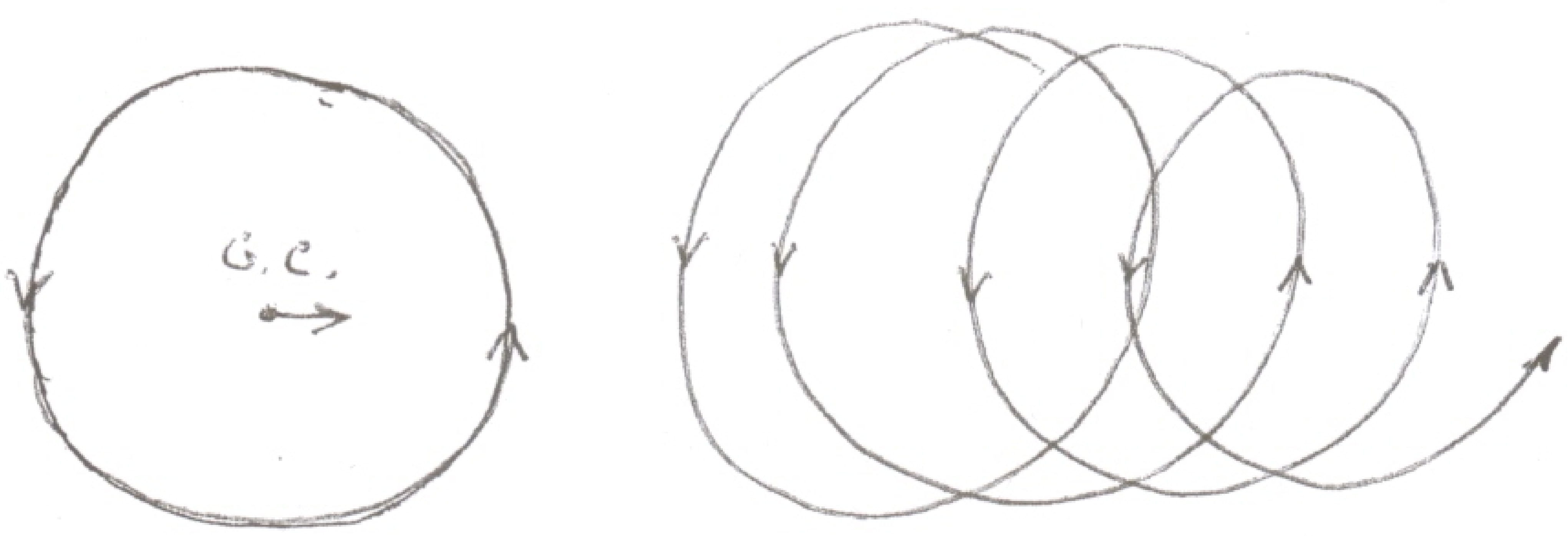
$$+ \frac{cM/\tau_i}{eBQ} \nabla \times \left[\frac{\nabla p_e}{n} + m\left(\frac{d\mathbf{u}}{dt} - \mathbf{L}_e \right) \right]$$

$$- \frac{1}{Q} \nabla \times \left\{ \left[\frac{\nabla(p_i + p_e)}{n} + M\left(\frac{d\mathbf{w}}{dt} - \mathbf{L}_i \right) + m\left(\frac{d\mathbf{u}}{dt} - \mathbf{L}_e \right) \right] \times \mathbf{b} \right\}$$

These extra terms include thermo-electric effects, the Biermann battery, the Eddington-Sweet effect, etc. which are all negligible under ordinary large-scale circumstances in astrophysical settings. But watch out for the small scales arising in tangential discontinuities, rapid reconnection, etc.

Compatibility of Newton and Maxwell

Consider a collisionless plasma, made up of equal numbers of electrons and singly charged ions.
Calculate the electron and ion motions using the guiding center approximation.



Write $\mathbf{u} = c\dfrac{\mathbf{E}\times\mathbf{B}}{\mathrm{B}^2}$, $\qquad \mathbf{E}_{\perp} = -\dfrac{\mathbf{u}\times\mathbf{B}}{c}$

The motion of the guiding center is

$$\mathbf{v} = \mathbf{u} + \frac{\frac{1}{2}\mathrm{M}\,\mathbf{w}_{\perp}^{2}c}{e\mathbf{B}^{4}}\mathbf{B}\times\nabla\tfrac{1}{2}\mathrm{B}^{2} + \frac{\mathrm{M}\,w_{\parallel}^{2}c}{e\mathbf{B}^{4}}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]$$

Note that

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} = -\frac{\mathbf{w}_{\perp}^{2}}{2\mathrm{B}^{4}}\mathbf{B}\{\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\}$$

Define

$$p_{\perp} = \sum_{1}\tfrac{1}{2}\mathrm{Mw}_{\perp}^{2}, \quad p_{\parallel} = \sum_{1}\mathrm{Mw}_{\parallel}^{2}.$$

The current density is

$$\mathbf{j}_{\perp} = \frac{c}{B^2}\mathbf{B} \times \left\{ \nabla p_{\perp} - \left[\frac{p_{\parallel} - p_{\perp}}{B^2} \right] (\mathbf{B} \cdot \nabla)\mathbf{B} + \mathrm{NM}\frac{d\mathbf{u}}{dt} \right\}$$

and Ampere's law becomes

$$\mathrm{NM}\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = -\nabla_{\perp}\left(p_{\perp} + \frac{\mathrm{B}^2}{8\pi} \right) + \frac{[(\mathbf{B} \cdot \nabla)\mathbf{B}]_{\perp}}{4\pi} \left\{ 1 + \frac{p_{\perp} - p_{\parallel}}{\mathrm{B}^2/4\pi} \right\}$$

So Ampere's law is automatically satisfied if the bulk velocity **u** satisfies Newton's equation.

See Phys. Rev. **107,** 924 (1957).

Chew-Goldberger-Low Approximation

Let L denote scale of plasma and field in the direction alongthe field. There are, then, four invariants.

$Lw_{\|} =$ constant

$AB =$ constant

$ALN =$ constant

$w^2_{\perp}/B =$ constant

where A is the characteristic cross section of a flux bundle and B the field.

So

$$\frac{d}{dt}\left(\frac{p_{\perp}}{NB}\right) = 0, \qquad \frac{d}{dt}\left(\frac{p_{\|}B}{N^3}\right) = 0$$

$$\frac{dp_{\perp}}{dt} = p_{\perp}\frac{d\ln NB}{dt} + \frac{p_{\|} - p_{\perp}}{\tau}$$

$$\frac{dp_{\|}}{dt} = p_{\|}\frac{d\ln (N^3/B)}{dt} - \frac{p_{\|} - p_{\perp}}{\tau}$$

ELECTRIC CIRCUIT ANALOG

It is asserted that the electric currents required by Ampere's law are subject to the familiar electric circuit equations.

MHD is equivalent to a Laboratory Electric Circuit.

However, in the laboratory circuit:

(a) Current paths have fixed connectivity.

(b) Current paths are fixed in the lab frame.

Whereas in MHD:

(a) Current paths and connections vary according to the dictates of Ampere's law,

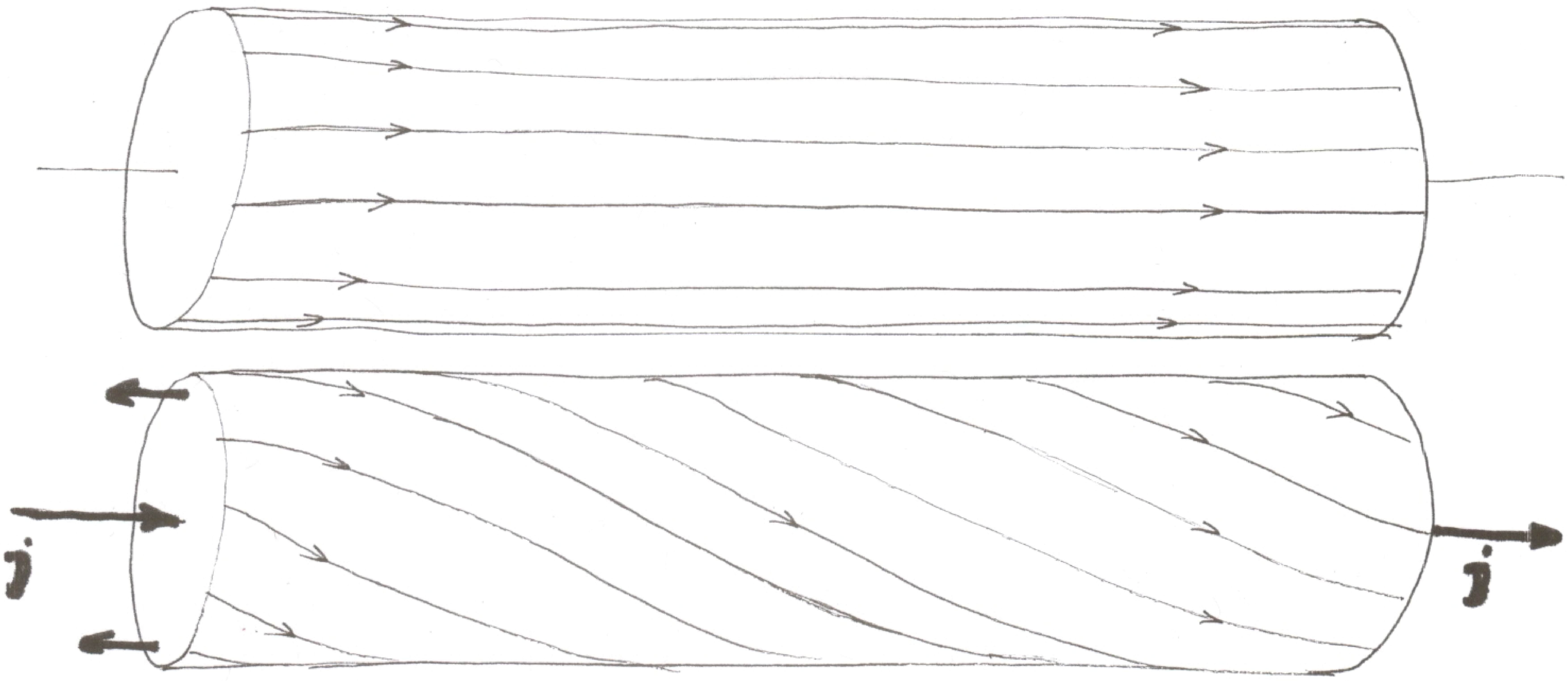
$$4\pi \boldsymbol{j} = c\nabla \times \boldsymbol{B},$$

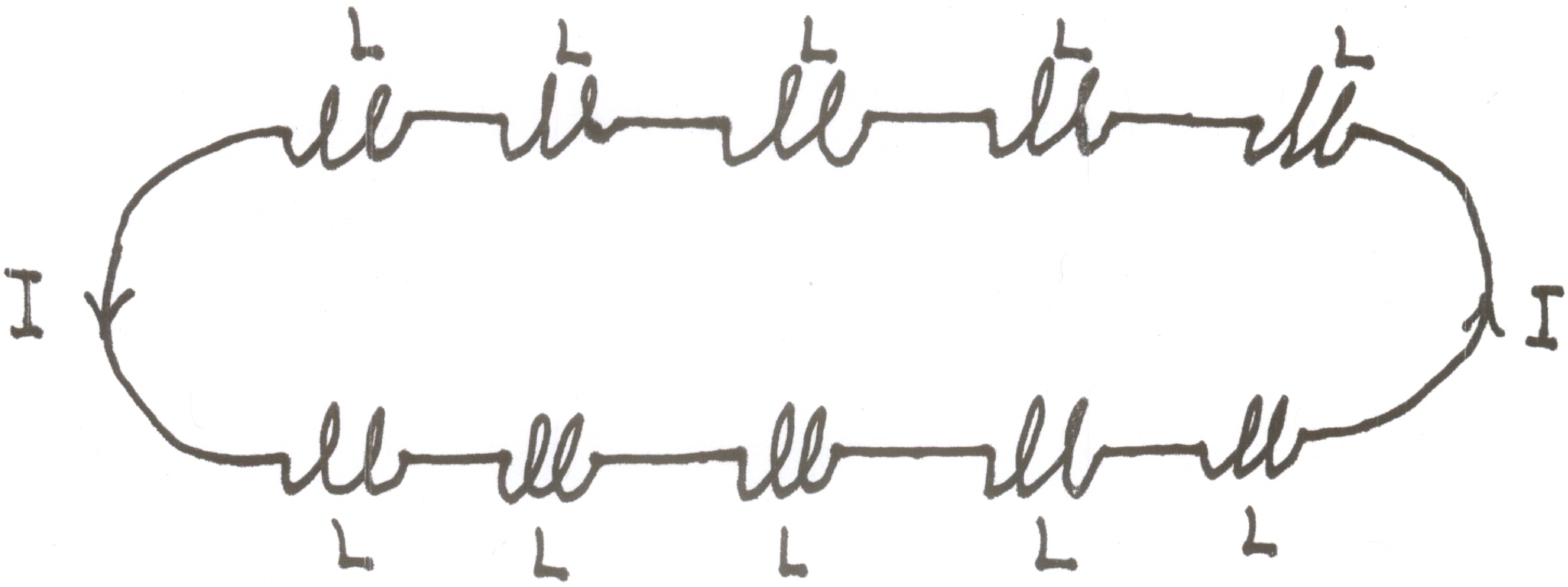
as the swirling fluid velocity $\boldsymbol{v}$ deforms $\boldsymbol{B}$,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}).$$

(b) The current flows in the moving frame of reference of $\boldsymbol{v}$, in which $\boldsymbol{E}' = 0$, so there are no inductive *emf*'s applied to $\boldsymbol{j}$.

There is no electric circuit analog in time-dependent MHD.

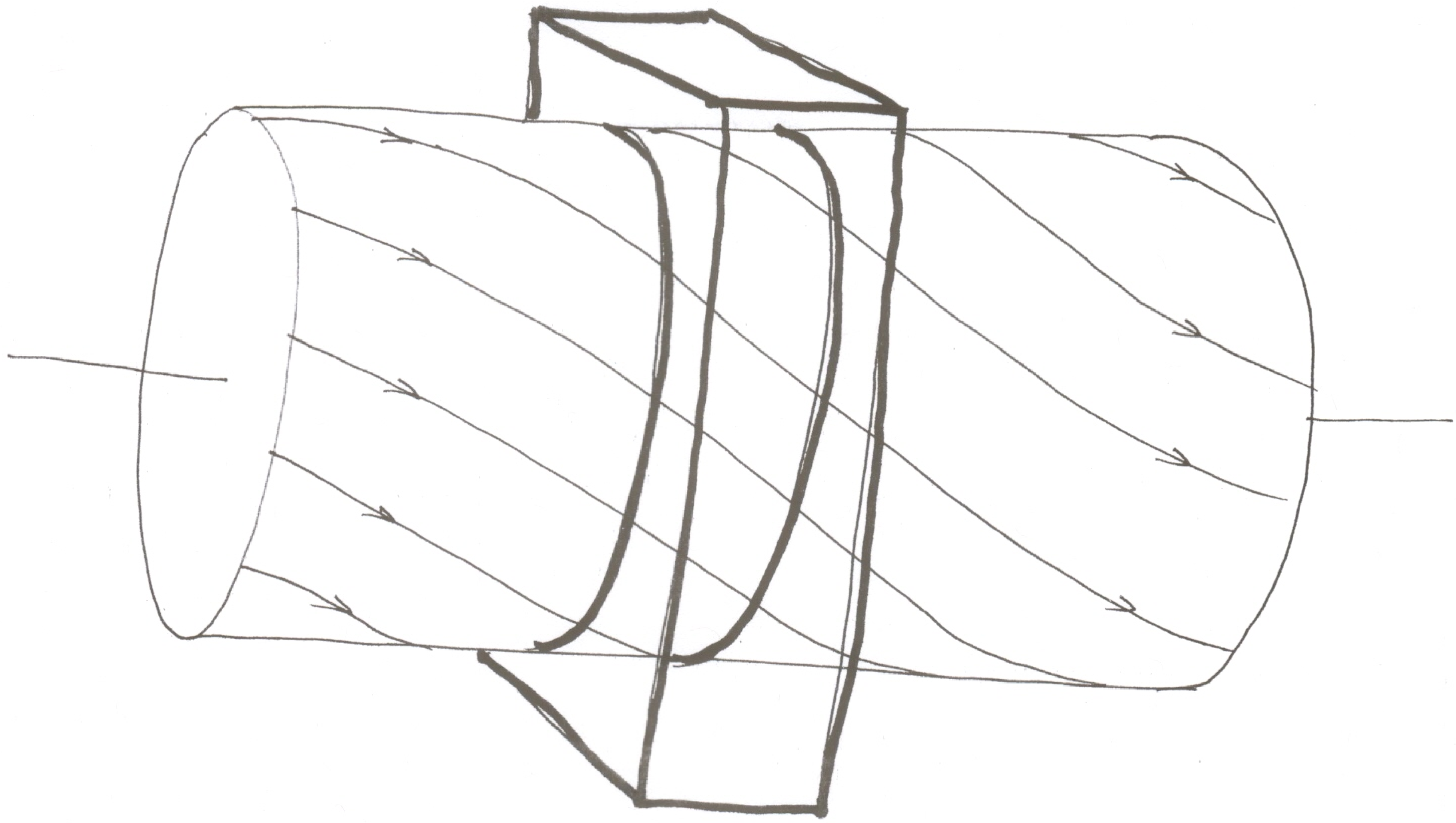


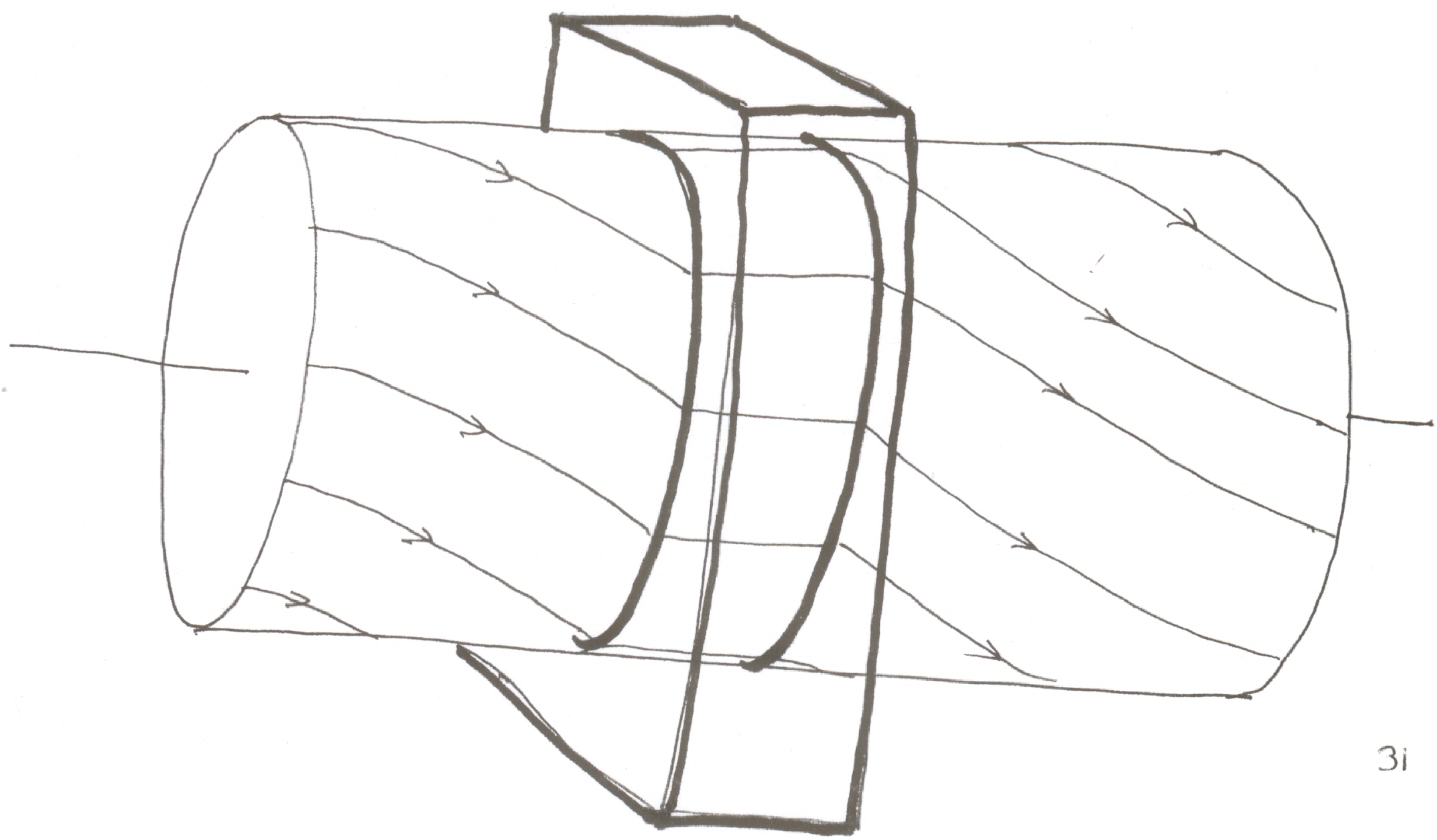


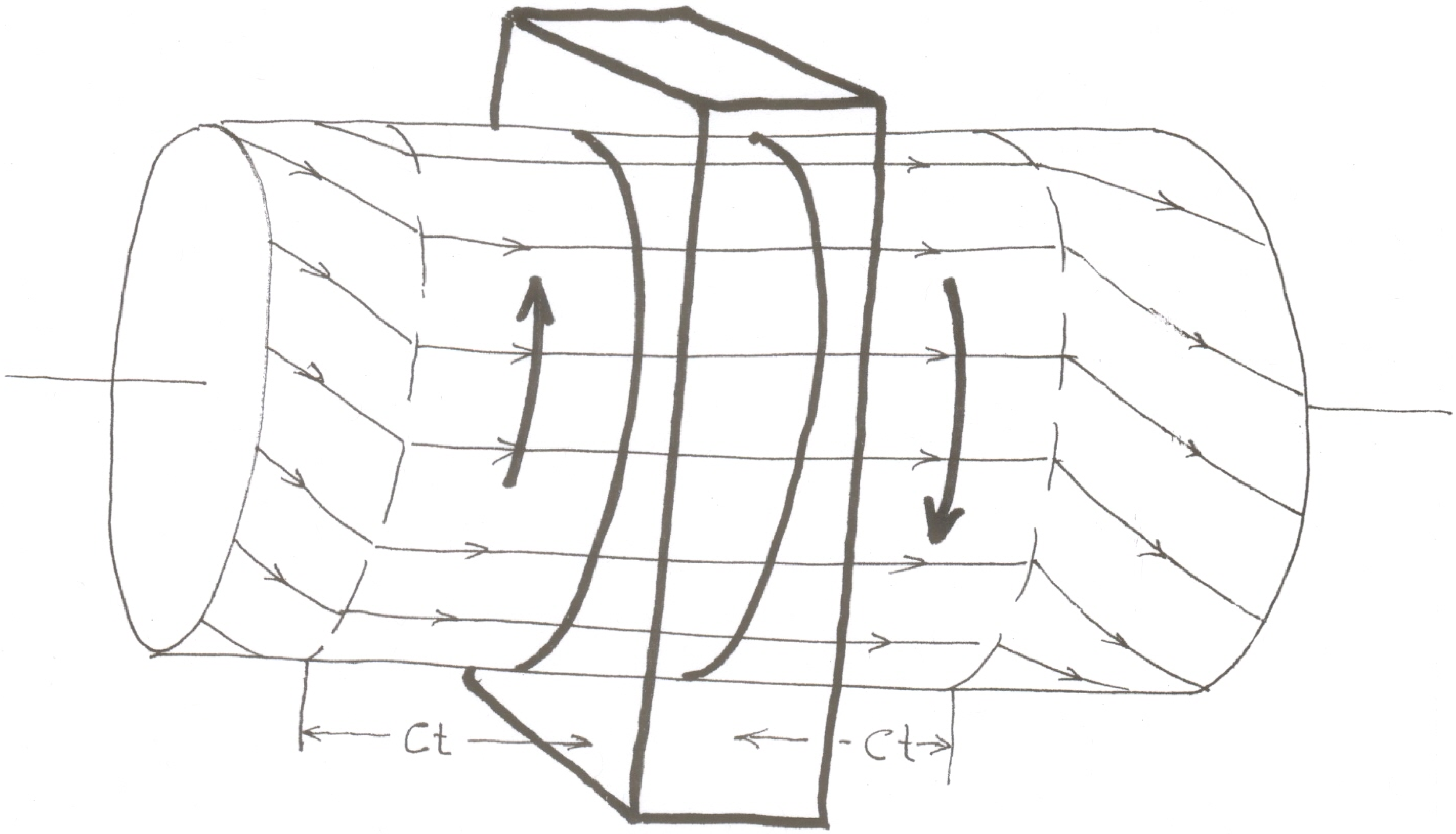
I = Total current

L = Inductance per unit length

$\frac{1}{2}LI^2$ = Magnetic energy per unit length







(c)