HYDRODYNAMICS
And
MAGNETO HYDRODYNAMICS

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PRELIMINARY REMARKS

Concerning hydrodynamics & magnetohydro dynamics in nature, where no one applies external electrical potentials.

Basic Point:

The dynamics of gases and magnetized plasmas is described by the equations of Newton and Maxwell

Consider the large-scale bulk motion of gases, plasmas, and magnetic field.

Consider the necessary and sufficient conditions for applicability of hydrodynamics (HD) to the bulk motion within a cloud of independently moving particles.

Imagine an infinite space filled with particles, each particle moving freely and independently along a straight path with its own arbitrary constant velocity u_i.

$$x_i(t) = x_i(0) + u_i t .$$

The initial mean particle density and/or the distribution of individual particle velocities is nonuniform on a scale L, so that the subsequent particle density $N(x_k, t)$ varies with time on the same scale.

What are the necessary and sufficient conditions that the bulk motion v_i is described by the continuum hydrodynamic equations?

An obvious necessary condition is that there are enough particles that the local mean particle density $N(x_i, t)$ is statistically well defined.

The local particle density $N(x_i, t)$ is defined as the mean over some small scale l ($<<L$).

Hence, it is necessary that

$$Nl^3 >> 1,$$

where l is chosen sufficiently small that the difference equations on the grid spacing l provide a good approximation to the differential equations. For most purposes it is sufficient that $l \leq 10^{-3} L$.

Smaller l may be needed to treat the build up of shock fronts and singular current sheets, of course.

So, given enough particles, the local density is well defined, and, therefore , the local bulk velocity, momentum density, kinetic energy density, etc, are all statistically well defined.

THEN ASSUME THAT THERE IS CONSERVATION OF PARTICLES, MOMENTUM, AND ENERGY.

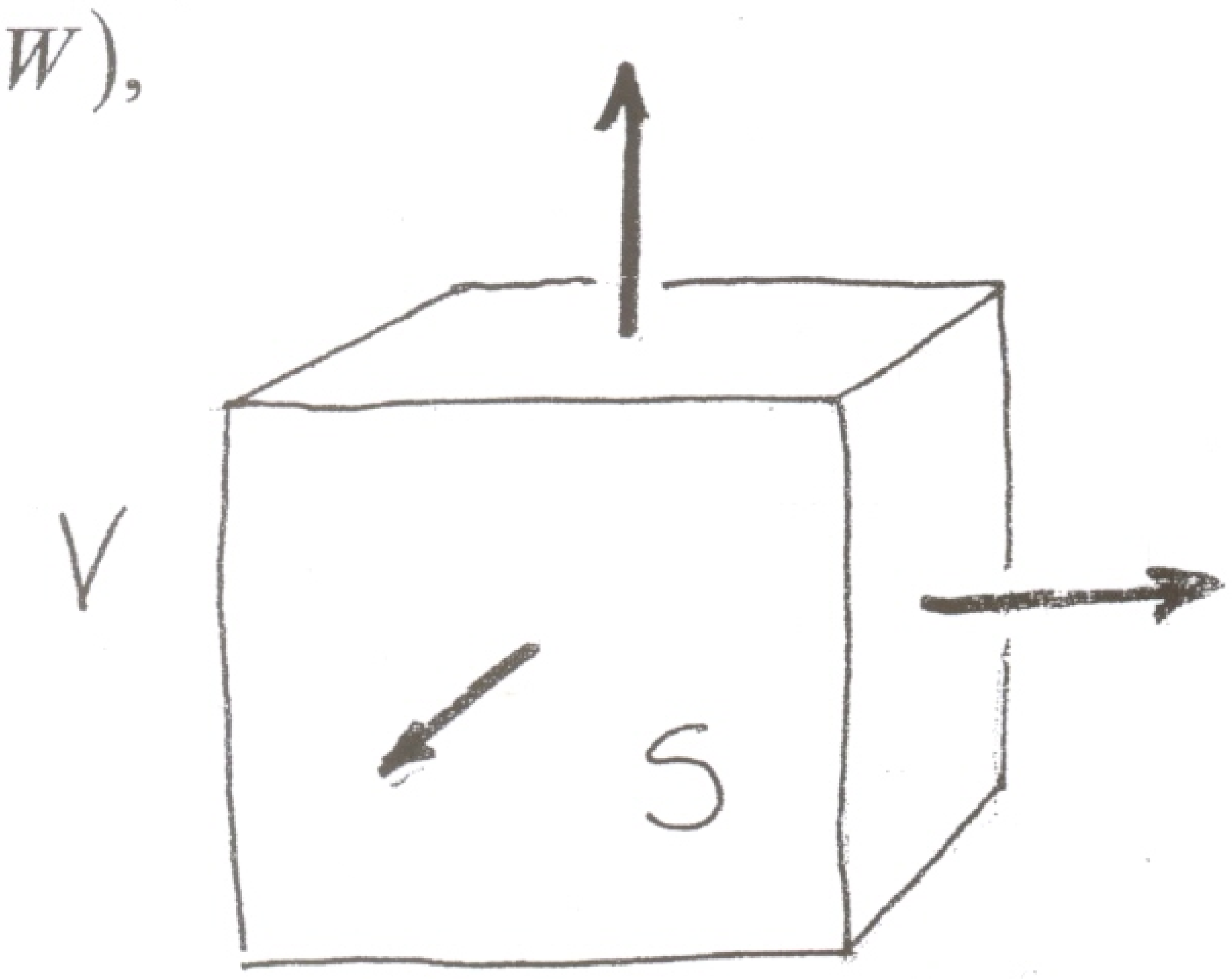
As we shall see, the familiar equations of HD are the result.

Note that the time derivative of the density W of some conserved quantity is equal to the negative divergence of the flux of W.

$$\int dV \frac{\partial W}{\partial t} = -\int d\mathbf{S} \cdot \mathbf{v}W$$

$$= -\int dV \nabla \cdot (\mathbf{v}W),$$

by Gauss's theorem.



Hence

$$\frac{\partial W}{\partial t} = -\nabla \cdot (\mathbf{v}W).$$

This is the mathematical statement needed to treat the three conserved quantities: particles, momentum, and energy.

Let u_i = velocity of individual particle with

$$u_i = v_i + w_i$$

v_i = local mean bulk velocity.

w_i = thermal velocity relative to mean.

Compute mean over local volume $V = l^3$

$$N = \frac{1}{V}\sum_V \ , Nv_i = \frac{1}{V}\sum_V u_i = \frac{1}{V}\sum_V (v_i + w_i)$$

$$\sum_v w_i = 0$$

Particle density N, particle flux $\sum_V Nu_i = Nv_i$

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x_k} Nv_k, \qquad \frac{dN}{dt} = -N\frac{\partial v_k}{\partial x_k}$$

Momentum density

$$\frac{1}{V}\sum_V Mu_i = NMv_i$$

Flux of momentum density

$$\frac{1}{V}\sum_V Mu_i u_j = \frac{1}{V}\sum_V Mv_i v_j + \frac{1}{V}\sum_V Mw_i w_j$$

Flux of momentum density

$$NMv_i v_j + p_{ij}, \qquad p_{ij} = \frac{1}{V}\sum_V Mw_i w_j$$

p_{ij} is pressure tensor = flux of momentum density carried by thermal motions.

The time rate of change of the momentum density is equal to the negative divergence of the flux of momentum density.

$$\frac{\partial}{\partial t} NMv_i = -\frac{\partial}{\partial x_j} NMv_i v_j - \frac{\partial}{\partial x_i} p_{ij}$$

Using the equation for conservation of particles, this reduces to the familiar Euler equation,

$$NM\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p_{ij}}{\partial x_j},$$

i.e Newton's equation of motion, recognizing that the momentum flux p_{ij} is equivalent to a force.

If an external force F_i (dynes/cm^3) is introduced, the momentum equation becomes

$$NM\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p_{ij}}{\partial x_j} + F_i$$

Density of momentum flux is

$$\frac{1}{V}\sum_V M u_i u_j = N M v_i v_j + p_{ij}$$

The flux of the density of momentum flux is

$$\frac{1}{V}\sum_V M u_i u_j u_k = \frac{1}{V}\sum_V M(v_i + w_i)(v_j + w_j)(v_k + w_k),$$

$$= N M v_i v_j v_k + v_i p_{jk} + v_j p_{ik} + v_k p_{ij} + T_{ijk},$$

with

$$T_{ijk} = \frac{1}{V}\sum_V M w_i w_j w_k$$

representing the heat flow tensor, which is the flux of p_{ij} transported by thermal motions.

Conservation of energy requires that

$$\frac{\partial}{\partial t}\frac{1}{V}\Sigma M u_i u_j = -\frac{\partial}{\partial x_k}\frac{1}{V}\Sigma M u_i u_j u_k,$$

reducing to

$$\frac{\partial}{\partial t}\left(NMv_i v_j + p_{ij}\right) = -\frac{\partial}{\partial x_k}\left(NMv_i v_j v_k + v_i p_{jk} + v_j p_{ik} + v_k p_{ij} + T_{ijk}\right).$$

The momentum equation can then be used to eliminate $\partial NMv_i v_j / \partial t$, so that

$$\frac{dp_{ij}}{dt} = \frac{\partial p_{ij}}{\partial t} + v_k \frac{\partial p_{ij}}{\partial x_k}$$

$$= -p_{jk} \frac{\partial v_i}{\partial x_k} - p_{ik} \frac{\partial v_j}{\partial x_k} - p_{ij} \frac{\partial v_k}{\partial x_k} - \frac{\partial T_{ijk}}{\partial x_k}.$$

Now there are circumstances where there is energy input from a radiation field, or from the dissipation of plasma waves or Alfven waves. To include an additional heat input, introduce the heat source S_{ij} on the right hand side of the equation, where S_{ij} is constructed to represent the appropriate heat input.

$$\frac{dp_{ij}}{dt} = -p_{jk} \frac{\partial v_i}{\partial x_k} - p_{ik} \frac{\partial v_j}{\partial x_k} - p_{ij} \frac{\partial v_k}{\partial x_k} - \frac{\partial T_{ijk}}{\partial x_k} + S_{ij}$$

The additional heat input S_{ij} often plays an essential role in the dynamics, as, for instance, in the solar wind, where the extreme supersonic velocity of the wind arises in large part from heat input from the dissipation of plasmas waves beyond the sonic point.

Consider some simple cases with $S_{ij} = \frac{\partial T_{ijk}}{\partial x_k} = 0$.

Adiabatic, collisionless

$$\frac{\partial v_1}{\partial x_1} = \frac{1}{\tau}, \quad \frac{\partial v_2}{\partial x_2} = \frac{\partial v_3}{\partial x_3} = 0, \quad \frac{\partial v_i}{\partial x_j} = 0 \quad \text{for } i \neq j$$

$$\frac{dp_{11}}{dt} = -\frac{3}{\tau} p_{11}, \quad \frac{dN}{dt} = -\frac{1}{\tau} N, \quad \frac{dp_{22}}{dt} = -\frac{1}{\tau} p_{22}$$

$$p_{11} \cong N^3, \quad p_{22}, \quad p_{33} \sim N$$

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = \frac{1}{\tau}, \quad \frac{\partial v_3}{\partial x_3} = 0$$

$$\frac{dp_{11}}{dt} = -\frac{4}{\tau} p_{11}, \quad \frac{dN}{dt} = -\frac{2}{\tau} N$$

$$p_{11}, p_{22} \sim N^2, \quad p_{33} \sim N$$

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = \frac{\partial v_3}{\partial x_3} = \frac{1}{\tau}$$

$$\frac{dp_{11}}{dt} = -\frac{5}{\tau} p_{11}, \qquad \frac{dN}{dt} = -\frac{3}{\tau} N$$

$$p_{11}, \quad p_{22}, \quad p_{33} \sim N^{\frac{5}{3}}$$

In the presence of collisions, add the scattering

$$\frac{dp_{11}}{dt} = -\frac{1}{\tau}(2p_{11} - p_{22} - p_{33}) + \ldots..,$$
$$\frac{dp_{22}}{dt} = -\frac{1}{\tau}(2p_{22} - p_{33} - p_{11}) + \ldots...,$$
$$\frac{dp_{33}}{dt} = -\frac{1}{\tau}(2p_{33} - p_{11} - p_{22}) + \ldots..,$$

For collision dominated gas $p_{11} = p_{22} = p_{33} \equiv p$

$$\frac{1}{\gamma - 1}\frac{dp}{dt} + \frac{\gamma}{\gamma - 1} p \frac{\partial v_k}{\partial x_k} = K\nabla^2 T$$

The off diagonal terms of p_{ij} can be represented by a suitable viscosity.

Magneto Hydrodynamics

MHD is based on the concept that the magnetic field is transported by the fluid

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \text{dissipation terms.}$$

Consider a gas with enough free electrons and ions that it cannot support any significant electric field $\mathbf{E}'$ in its own frame of reference.

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$$

If $\mathbf{E}'=0$, then

$$\mathbf{E} = -\frac{\mathrm{v}}{c} \times \mathbf{B}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

The result is MHD, regardless of the details

ELECTROMAGNETIC PRELIMINAIRES

Non relativistic ($v^2/c^2 << 1$) Lorentz transformation of $\boldsymbol{E}$ and $\boldsymbol{B}$.

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}, \boldsymbol{B}' = \boldsymbol{B} - \frac{\boldsymbol{v}\times\boldsymbol{E}}{c}.$$

$\boldsymbol{E}', \boldsymbol{B}'$ are the fields experienced in the reference frame with velocity $\boldsymbol{v}$ relative to the frame in which the fields are $\boldsymbol{E}$,$\boldsymbol{B}$, respectively.

A moving physical system experiences only the fields $\boldsymbol{E}', \boldsymbol{B}'$ in its own reference frame.

There is a magnetic field of about half a Gauss filling this room.

Is there an electric field in this room?

To determine the dynamical role of **E, B** consider Poynting's theorem for a collection of particles with mass distribution $\rho(\mathbf{r},t)$ and associated charge distribution $\delta(\mathbf{r},t)$. The individual particle has velocity **v** and

$$\mathbf{j} = \mathbf{v}\delta(\mathbf{r},t)$$

$$\rho\frac{d\mathbf{v}}{dt} = \delta\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)$$
$$= \frac{\mathbf{E}\nabla\cdot\mathbf{E}}{4\pi} + \mathbf{j}\times\mathbf{B}/c$$

Using Maxwell's equations, it can be shown that

$$\rho\frac{dv_i}{dt} = \frac{\partial M_{ij}}{\partial x_j} - \frac{\partial}{\partial t}\left(\frac{P_i}{c^2}\right)$$

or

$$\frac{\partial}{\partial t}\left(\rho v_i + \frac{P_i}{c^2}\right) = \frac{\partial M_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j}$$

where

$$M_{ij} = -\delta_{ij}\frac{E^2 + \mathrm{B}^2}{8\pi} + \frac{E_iE_j + B_iB_j}{4\pi}$$ Maxwell stress tensor

$$R_{ij} = -\rho v_i v_j$$ Reynolds stress tensor

$$\mathbf{P} = c\frac{\mathbf{E}\times\mathbf{B}}{4\pi}$$ Poynting vector