Chapter 5

Elements of Analytical Photogrammetry

5.1 Introduction, Concept of Image and Object Space

Photogrammetry is the science of obtaining reliable information about objects and of measuring and interpreting this information. The task of obtaining information is called data acquisition, a process we discussed at length in GS601, Chapter 2. Fig. 5.1(a) depicts the data acquisition process. Light rays reflected from points on the object, say from point A, form a divergent bundle which is transformed to a convergent bundle by the lens. The principal rays of each bundle of all object points pass through the center of the entrance and exit pupil, unchanged in direction. The front and rear nodal points are good approximations for the pupil centers.

Another major task of photogrammetry is concerned with reconstructing the object space from images. This entails two problems: geometric reconstruction (e.g. the position of objects) and radiometric reconstruction (e.g. the gray shades of a surface). The latter problem is relevant when photographic products are generated, such as orthophotos. Photogrammetry is mainly concerned with the geometric reconstruction. The object space is only partially reconstructed, however. With partial reconstruction we mean that only a fraction of the information recorded from the object space is used for its representation. Take a map, for example. It may only show the perimeter of buildings, not all the intricate details which make up real buildings.

Obviously, the success of reconstruction in terms of geometrical accuracy depends largely on the similarity of the image bundle compared to the bundle of principal rays that entered the lens during the instance of exposure. The purpose of camera calibration is to define an image space so that the similarity becomes as close as possible.

The geometrical relationship between image and object space can best be established by introducing suitable coordinate systems for referencing both spaces. We describe the coordinate systems in the next section. Various relationships exist between image and object space. In Table 5.1 the most common relationships are summarized, together with the associated photogrammetric procedures and the underlying mathematical models.

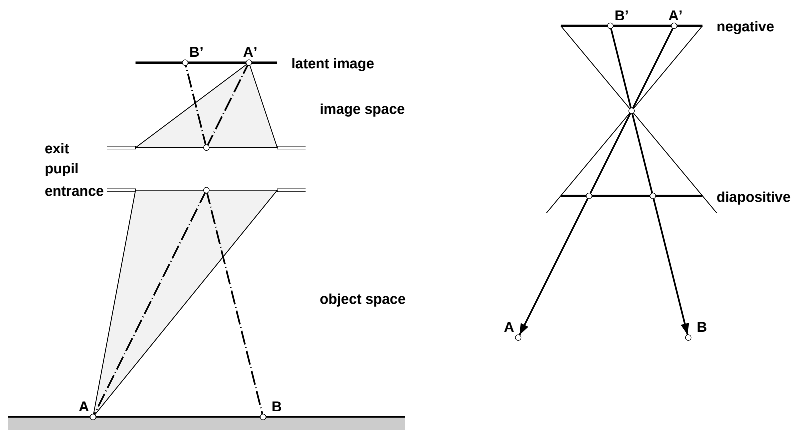

Figure 5.1: In (a) the data acquisition process is depicted. In (b) we illustrate the reconstruction process.

In this chapter we describe these procedures and the mathematical models, except aerotriangulation (block adjustment) which will be treated later. For one and the same procedure, several mathematical models may exist. They differ mainly in the degree of complexity, that is, how closely they describe physical processes. For example, a similarity transformation is a good approximation to describe the process of converting measured coordinates to photo-coordinates. This simple model can be extended to describe more closely the underlying measuring process. With a few exceptions, we will not address the refinement of the mathematical model.

5.2 Coordinate Systems

5.2.1 Photo-Coordinate System

The photo-coordinate system serves as the reference for expressing spatial positions and relations of the image space. It is a 3-D cartesian system with the origin at the perspective center. Fig. 5.2 depicts a diapositive with fiducial marks that define the fiducial center *FC*. During the calibration procedure, the offset between fiducial center and principal point of autocollimation, *PP*, is determined, as well as the origin of the radial distortion, *PS*. The x, y coordinate plane is parallel to the photograph and the positive $x-$axis points toward the flight direction.

Positions in the image space are expressed by point vectors. For example, point vector $\mathbf{p}$ defines the position of point P on the diapositive (see Fig. 5.2). Point vectors of positions on the diapositive (or negative) are also called *image vectors*. We have for point P

Table 5.1: Summary of the most important relationships between image and object space.

relationship between	*procedure*	*mathematical model*
measuring system and photo-coordinate system	interior orientation	2-D transformation
photo-coordinate system and object coordinate system	exterior orientation	collinearity eq.
photo-coordinate systems of a stereopair	relative orientation	collinearity eq. coplanarity condition
model coordinate system and object coordinate system	absolute orientation	7-parameter transformation
several photo-coordinate systems and object coordinate system	bundle block adjustment	collinearity eq.
several model coordinate systems and object coordinate system	independent model block adjustment	7 parameter transformation

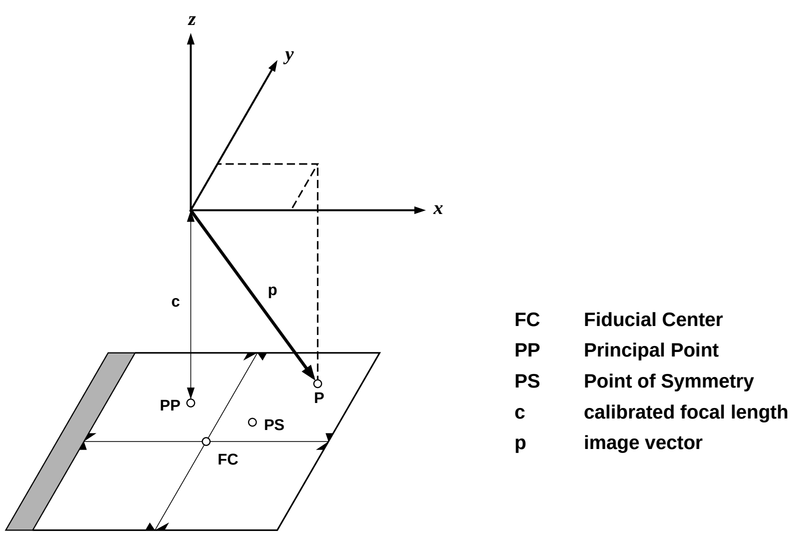

Figure 5.2: Definition of the photo-coordinate system.

$$\mathbf{p} = \begin{bmatrix} x_p \\ y_p \\ -c \end{bmatrix} \tag{5.1}$$

Note that for a diapositive the third component is negative. This changes to a positive value in the rare case a negative is used instead of a diapositive.

5.2.2 Object Space Coordinate Systems

In order to keep the mathematical development of relating image and object space simple, both spaces use 3-D cartesian coordinate systems. Positions of control points in object space are likely available in another coordinate systems, e.g. State Plane coordinates. It is important to convert any given coordinate system to a cartesian system before photogrammetric procedures, such as orientations or aerotriangulation, are performed.

5.3 Interior Orientation

We have already introduced the term *interior orientation* in the discussion about camera calibration (see GS601, Chapter 2), to define the metric characteristics of aerial cameras. Here we use the same term for a slightly different purpose. From Table 5.1 we conclude that the purpose of interior orientation is to establish the relationship between a measuring system[1] and the photo-coordinate system. This is necessary because it is not possible to measure photo-coordinates directly. One reason is that the origin of the photo-coordinate system is only mathematically defined; since it is not visible it cannot coincide with the origin of the measuring system.

Fig. 5.3 illustrates the case where the diapositive to be measured is inserted in the measuring system whose coordinate axis are xm, ym. The task is to determine the transformation parameters so that measured points can be transformed into photo-coordinates.

5.3.1 Similarity Transformation

The most simple mathematical model for interior orientation is a similarity transformation with the four parameters: translation vector **t**, scale factor s, and rotation angle α.

$$xf = s(xm\,\cos(\alpha) - ym\,\sin(\alpha)) - xt \quad (5.2)$$
$$yf = s(xm\,\sin(\alpha) + ym\,\cos(\alpha)) - yt \quad (5.3)$$

These equations can also be written in the following form:

$$xf = a_{11}xm - a_{12}ym - xt \quad (5.4)$$
$$yf = a_{12}xm + a_{11}ym - yt \quad (5.5)$$

If we consider a_{11}, a_{12}, xt, yt as parameters, then above equations are linear in the parameters. Consequently, they can be directly used as observation equations for a least-squares adjustment. Two observation equations are formed for every point known in

[1]Measuring systems are discussed in the next chapter.

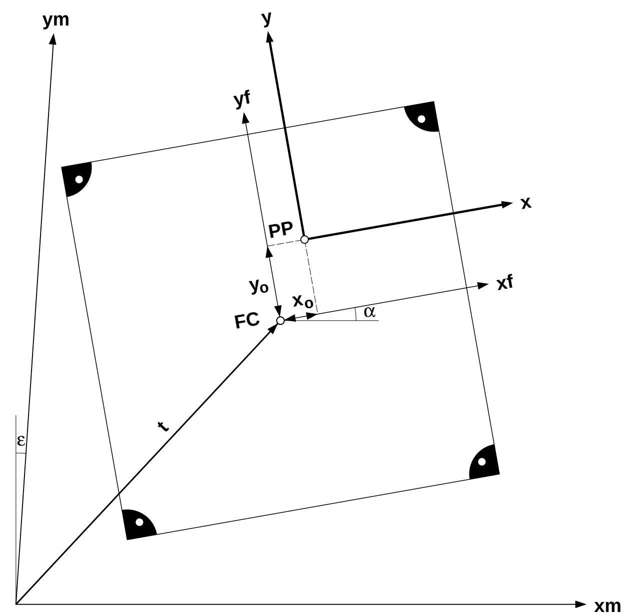

Figure 5.3: Relationship between measuring system and photo-coordinate system.

both coordinate systems. Known points in the photo-coordinate system are the fiducial marks. Thus, computing the parameters of the interior orientation amounts to measuring the fiducial marks (in the measuring system).

Actually, the fiducial marks are known with respect to the fiducial center. Therefore, the process just described will determine parameters with respect to the *fiducial coordinate system xf, yf*. Since the origin of the photo-coordinate system is known in the fiducial system (x_0, y_0), the photo-coordinates are readily obtained by the translation

$$x = xf - x_0 \tag{5.6}$$

$$y = yf - y_0 \tag{5.7}$$

5.3.2 Affine Transformation

The affine transformation is an improved mathematical model for the interior orientation because it more closely describes the physical reality of the measuring system. The parameters are two scale factors s_x, s_y, a rotation angle α, a skew angle ϵ, and a translation vector $\mathbf{t} = [xt, yt]^T$. The measuring system is a manufactured product and, as such, not perfect. For example, the two coordinate axis are not exactly rectangular, as indicated in Fig. 5.3(b). The skew angle expresses the nonperpendicularity. Also, the scale is different between the the two axis.

We have

$$xf = a_{11}xm + a_{12}ym - xt \tag{5.8}$$
$$yf = a_{21}xm + a_{22}ym - yt \tag{5.9}$$

where

$$\begin{array}{ll} a_{11} & s_x(\cos(\alpha - \epsilon\sin(\alpha)) \\ a_{12} & -s_y(\sin(\alpha)) \\ a_{21} & s_x(\sin(\alpha + \epsilon\cos(\alpha)) \end{array}$$

Eq. 4.8 and 5.9 are also linear in the parameters. Like in the case of a similarity transformation, these equations can be directly used as observation equations. With four fiducial marks we obtain eight equations leaving a redundancy of two.

5.3.3 Correction for Radial Distortion

As discussed in GS601 Chapter 2, radial distortion causes off-axial points to be radially displaced. A positive distortion increases the lateral magnification while a negative distortion reduces it.

Distortion values are determined during the process of camera calibration. They are usually listed in tabular form, either as a function of the radius or the angle at the perspective center. For aerial cameras the distortion values are very small. Hence, it suffices to linearly interpolate the distortion. Suppose we want to determine the distortion for image point x_p, y_p. The radius is $r_p = (x_p^2 + y_p^2)^{1/2}$. From the table we obtain the distortion dr_i for $r_i < r_p$ and dr_j for $r_j > r_p$. The distortion for r_p is interpolated

$$dr_p = \frac{(dr_j - dr_i)\, r_p}{(r_j - r_i)} \tag{5.10}$$

As indicated in Fig. 5.4 the corrections in x- and y-direction are

$$dr_x = \frac{x_p}{r_p} dr_p \tag{5.11}$$
$$dr_y = \frac{y_p}{r_p} dr_p \tag{5.12}$$

Finally, the photo-coordinates must be corrected as follows:

$$x_p = x_p - dr_x = x_p(1 - \frac{dr_p}{r_p}) \tag{5.13}$$
$$y_p = y_p - dr_y = y_p(1 - \frac{dr_p}{r_p}) \tag{5.14}$$

The radial distortion can also be represented by an odd-power polynomial of the form

$$dr = p_0\, r + p_1\, r^3 + p_2\, r^5 + \cdots \tag{5.15}$$

The coefficients p_i are found by fitting the polynomial curve to the distortion values. Eq. 5.15 is a linear observation equation. For every distortion value, an observation equation is obtained.

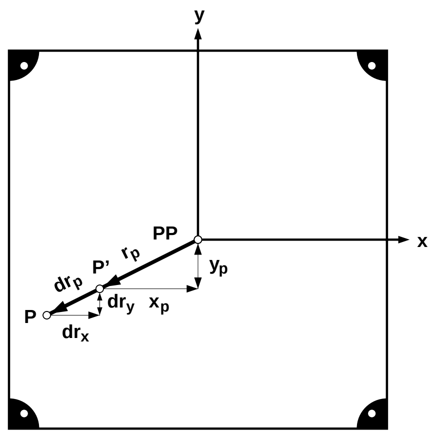

Figure 5.4: Correction for radial distortion.

In order to avoid numerical problems (ill-conditioned normal equation system), the degree of the polynom should not exceed nine.

5.3.4 Correction for Refraction

Fig. 5.5 shows how an oblique light ray is refracted by the atmosphere. According to Snell's law, a light ray is refracted at the interface of two different media. The density differences in the atmosphere are in fact different media. The refraction causes the image to be displayed outwardly, quite similar to a positive radial distortion.

The radial displacement caused by refraction can be computed by

$$dref = K(r + \frac{r^3}{c^2}) \tag{5.16}$$

$$K = \left(\frac{2410\, H}{H^2 - 6\, H + 250} - \frac{2410\, h^2}{(h^2 - 6\, h + 250)H}\right) 10^{-6} \tag{5.17}$$

These equations are based on a model atmosphere defined by the US Air Force. The flying height H and the ground elevation h must be in units of kilometers.

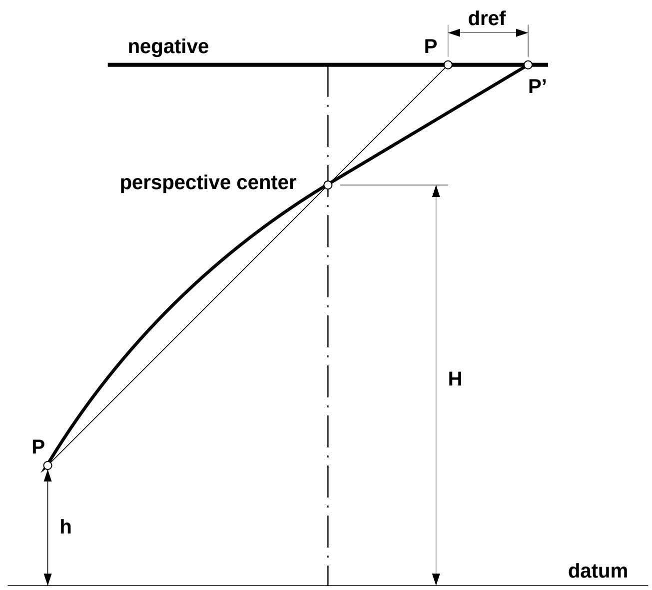

Figure 5.5: Correction for refraction.

5.3.5 Correction for Earth Curvature

As mentioned in the beginning of this Chapter, the mathematical derivation of the relationships between image and object space are based on the assumption that for both spaces, 3-D cartesian coordinate systems are employed. Since ground control points may not directly be available in such a system, they must first be transformed, say from a State Plane coordinate system to a cartesian system.

The X and Y coordinates of a State Plane system are cartesian, but not the elevations. Fig. 5.6 shows the relationship between elevations above a datum and elevations in the 3-D cartesian system. If we approximate the datum by a sphere, radius $R = 6372.2$ km, then the radial displacement can be computed by

$$\textit{dearth} = \frac{r^3\,(H - Z_P)}{2\,c^2\,R} \tag{5.18}$$

Like radial distortion and refraction, the corrections in $x-$ and y-direction is readily determined by Eq. 4.13 and 5.14. Strictly speaking, the correction of photo-coordinates due to earth curvature is not a refinement of the mathematical model. It is much better to eliminate the influence of earth curvature by transforming the object space into a 3-D cartesian system before establishing relationships with the ground system. This is always possible, except when compiling a map. A map, generated on an analytical plotter, for example, is most likely plotted in a State Plane coordinate system. That is,

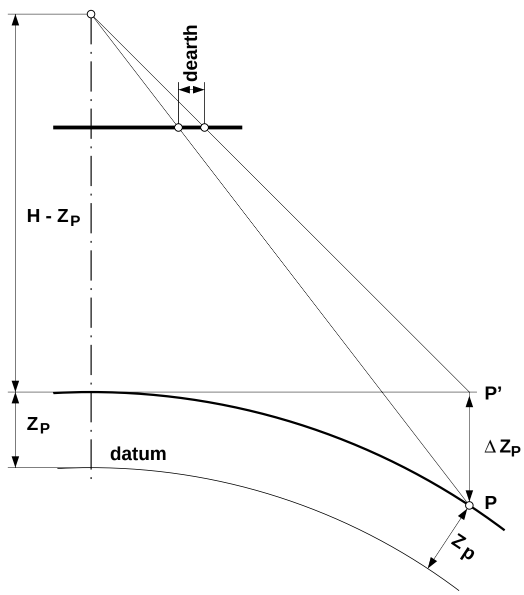

Figure 5.6: Correction of photo-coordinates due to earth curvature.

the elevations refer to the datum and not to the XY plane of the cartesian coordinate system. It would be quite awkward to produce the map in the cartesian system and then transform it to the target system. Therefore, during map compilation, the photo-coordinates are "corrected" so that conjugate bundle rays intersect in object space at positions related to reference sphere.

5.3.6 Summary of Computing Photo-Coordinates

We summarize the main steps necessary to determine photo-coordinates. The process to correct them for systematic errors, such as radial distortion, refraction and earth curvature is also known as *image refinement*. Fig. 5.7 depicts the coordinate systems involved, an imaged point P, and the correction vectors *dr, dref, dearth*.

1. Insert the diapositive into the measuring system (e.g. comparator, analytical plotter) and measure the fiducial marks in the machine coordinate system *xm, ym*. Compute the transformation parameters with a similarity or affine transformation. The transformation establishes a relationship between the measuring system and the fiducial coordinate system.

2. Translate the fiducial system to the photo-coordinate system (Eqs. 4.6 and 5.7).

3. Correct photo-coordinates for radial distortion. The radial distortion dr_p for point

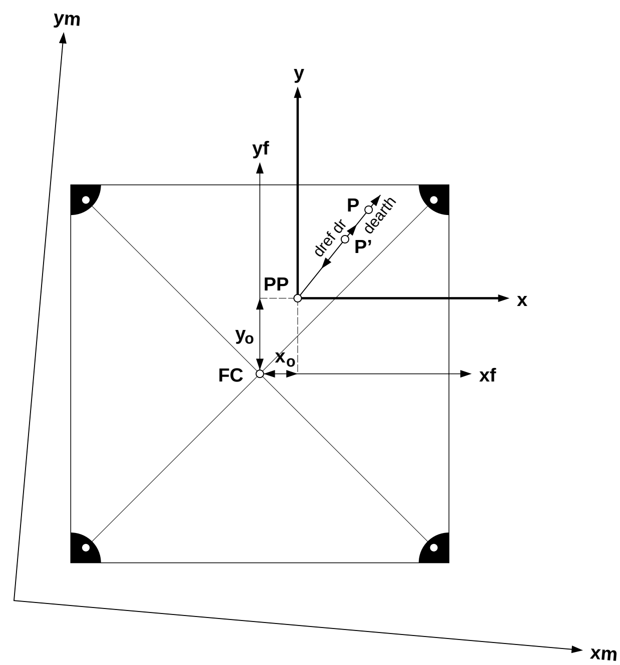

Figure 5.7: Interior orientation and image refinement.

P is found by linearly interpolating the values given in the calibration protocol (Eq. 5.10).

4. Correct the photo-coordinates for refraction, according to Eqs. 4.16 and 5.17. This correction is negative. The displacement caused by refraction is a functional relationship of ***dref*** $= f(H, h, r, c)$. With a flying height $H = 2,000\ m$, elevation above ground $h = 500\ m$ we obtain for a wide angle camera ($c \approx 0.15\ m$) a correction of $-4\,\mu m$ for $r = 130$ mm. An extreme example is a superwide angle camera, $H = 9,000$ m, $h = 500$ m, where ***dref*** $= -34\ \mu m$ for the same point.

5. Correct for earth curvature only if the control points (elevations) are not in a cartesian coordinate system or if a map is compiled. Using the extreme example as above, we obtain ***dearth*** $= 65\ \mu$m. Since this correction has the opposite sign of the refraction, the combined correction for refraction and earth curvature would be ***dcomb*** $= 31\ \mu m$. The correction due to earth curvature is larger than the correction for refraction.

5.4 Exterior Orientation

Exterior orientation is the relationship between image and object space. This is accomplished by determining the camera position in the object coordinate system. The camera position is determined by the location of its perspective center and by its attitude, expressed by three independent angles.

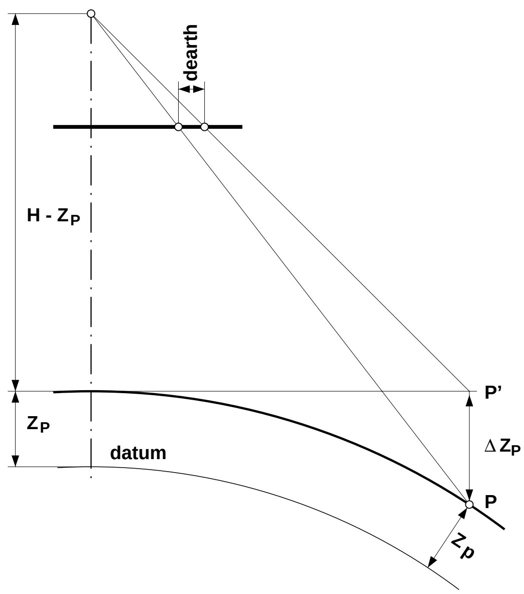

Figure 5.8: Exterior Orientation.

The problem of establishing the six orientation parameters of the camera can conveniently be solved by the collinearity model. This model expresses the condition that the perspective center C, the image point P_i, and the object point P_o, must lie on a straight line (see Fig. 5.8). If the exterior orientation is known, then the image vector $\mathbf{p}_i$ and the vector $\mathbf{q}$ in object space are collinear:

$$\mathbf{p}_i = \frac{1}{\lambda}\mathbf{q} \tag{5.19}$$

As depicted in Fig. 5.8, vector $\mathbf{q}$ is the difference between the two point vectors $\mathbf{c}$ and $\mathbf{p}$. For satisfying the collinearity condition, we rotate and scale $\mathbf{q}$ from object to image space. We have

$$\mathbf{p}_i = \frac{1}{\lambda}\mathbf{R}\,\mathbf{q} = \frac{1}{\lambda}\mathbf{R}\,(\mathbf{p} - \mathbf{c}) \tag{5.20}$$

with $\mathbf{R}$ an orthogonal rotation matrix with the three angles ω, ϕ and κ:

$$\mathbf{R} = \begin{vmatrix} \cos\phi\cos\kappa & -\cos\phi\sin\kappa & \sin\phi \\ \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & -\sin\omega\cos\phi \\ \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa & \cos\omega\cos\phi \end{vmatrix} \tag{5.21}$$

Eq. 5.20 renders the following three coordinate equations.

$$x = \frac{1}{\lambda}(X_P - X_C)r_{11} + (Y_P - Y_C)r_{12} + (Z_P - Z_C)r_{13} \tag{5.22}$$

$$y = \frac{1}{\lambda}(X_P - X_C)r_{21} + (Y_P - Y_C)r_{22} + (Z_P - Z_C)r_{23} \tag{5.23}$$

$$-c = \frac{1}{\lambda}(X_P - X_C)r_{31} + (Y_P - Y_C)r_{32} + (Z_P - Z_C)r_{33} \tag{5.24}$$

By dividing the first by the third and the second by the third equation, the scale factor $\frac{1}{\lambda}$ is eliminated leading to the following two collinearity equations:

$$x = -c\frac{(X_P - X_C)r_{11} + (Y_P - Y_C)r_{12} + (Z_P - Z_C)r_{13}}{(X_P - X_C)r_{31} + (Y_P - Y_C)r_{32} + (Z_P - Z_C)r_{33}} \tag{5.25}$$

$$y = -c\frac{(X_P - X_C)r_{21} + (Y_P - Y_C)r_{22} + (Z_P - Z_C)r_{23}}{(X_P - X_C)r_{31} + (Y_P - Y_C)r_{32} + (Z_P - Z_C)r_{33}} \tag{5.26}$$

with:

$$\mathbf{p}_i = \begin{bmatrix} x \\ y \\ -f \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

The six parameters: $X_C, Y_C, Z_C, \omega, \phi, \kappa$ are the unknown elements of exterior orientation. The image coordinates x, y are normally known (measured) and the calibrated focal length c is a constant. Every measured point leads to two equations, but also adds three other unknowns, namely the coordinates of the object point (X_P, Y_P, Z_P). Unless the object points are known (control points), the problem cannot be solved with only one photograph.

The collinearity model as presented here can be expanded to include parameters of the interior orientation. The number of unknowns will be increased by three[2]. This combined approach lets us determine simultaneously the parameters of interior and exterior orientation of the cameras.

There are only limited applications for single photographs. We briefly discuss the computation of the exterior orientation parameters, also known as single photograph resection, and the computation of photo-coordinates with known orientation parameters. Single photographs cannot be used for the main task of photogrammetry, the reconstruction of object space. Suppose we know the exterior orientation of a photograph. Points in object space are not defined, unless we also know the scale factor $1/\lambda$ for every bundle ray.

[2]Parameters of interior orientation: position of principal point and calibrated focal length. Additionally, three parameters for radial distortion and three parameters for tangential distortion can be added.

5.4.1 Single Photo Resection

The position and attitude of the camera with respect to the object coordinate system (exterior orientation of camera) can be determined with help of the collinearity equations. Eqs. 5.26 and 4.27 express measured quantities[3] as a function of the exterior orientation parameters. Thus, the collinearity equations can be directly used as observation equations, as the following functional representation illustrates.

$$x, y = f(\underbrace{X_C, Y_C, Z_C, \omega, \phi, \kappa}_{\text{exterior orientation}}, \underbrace{X_P, Y_P, Z_P}_{\text{object point}}) \tag{5.27}$$

For every measured point two equations are obtained. If three control points are measured, a total of 6 equations is formed to solve for the 6 parameters of exterior orientation.

The collinearity equations are not linear in the parameters. Therefore, Eqs. 4.25 and 5.26 must be linearized with respect to the parameters. This also requires approximate values with which the iterative process will start.

5.4.2 Computing Photo Coordinates

With known exterior orientation elements photo-coordinates can be easily computed from Eqs. 4.25 and 5.26. This is useful for simulation studies where synthetic photo-coordinates are computed.

Another application for the direct use of the collinearity equations is the real-time loop of analytical plotters where photo-coordinates of ground points or model points are computed after relative or absolute orientation (see next chapter, analytical plotters).

5.5 Orientation of a Stereopair

5.5.1 Model Space, Model Coordinate System

The application of single photographs in photogrammetry is limited because they cannot be used for reconstructing the object space. Even though the exterior orientation elements may be known it will not be possible to determine ground points unless the scale factor of every bundle ray is known. This problem is solved by exploiting stereopsis, that is by using a second photograph of the same scene, taken from a different position.

Two photographs with different camera positions that show the same area, at least in part, is called a *stereopair*. Suppose the two photographs are oriented such that *conjugate points* (corresponding points) intersect. We call this intersection space *model space*. In order for expressing relationships of this model space we introduce a reference system, the *model coordinate system*. This system is 3-D and cartesian. Fig. 5.9 illustrates the concept of model space and model coordinate system.

Introducing the model coordinate system requires the definition of its spatial position (origin, attitude), and its scale. These are the seven parameters we have encountered

[3]We assume that the photo-coordinates are measured. In fact they are derived from measured machine coordinates. The correlation caused by the transformation is neglected.

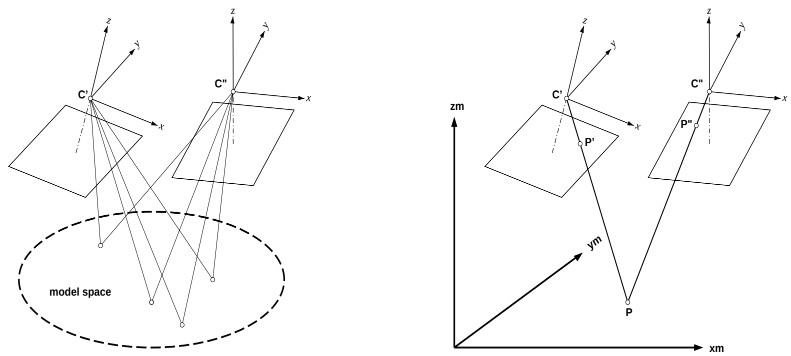

Figure 5.9: The concept of model space (a) and model coordinate system (b).

in the transformation of 3-D cartesian systems. The decision on how to introduce the parameters depends on the application; one definition of the model coordinate system may be more suitable for a specific purpose than another. In the following subsections, different definitions will be discussed.

Now the orientation of a stereopair amounts to determining the exterior orientation parameters of both photographs, with respect to the model coordinate system. From single photo resection, we recall that the collinearity equations form a suitable mathematical model to express the exterior orientation. We have the following functional relationship between observed photo-coordinates and orientation parameters:

$$x, y = f(\underbrace{X'_C, Y'_C, Z'_C, \omega', \phi', \kappa'}_{\text{ext. or}'}, \underbrace{X''_C, Y''_C, Z''_C, \omega'', \phi'', \kappa''}_{\text{ext. or}''}, \underbrace{X_1, Y_1, Z_1}_{\text{mod. pt 1}}, \cdots, \underbrace{X_n, Y_n, Z_n}_{\text{mod. pt n}}) \tag{5.28}$$

where f refers to Eqs. 4.25 and 5.26. Every point measured in one photo-coordinate system renders two equations. The same point must also be measured in the second photo-coordinate system. Thus, for one model point we obtain 4 equations, or $4\,n$ equations for n object points. On the other hand, n unknown model points lead to $3\,n$ parameters, or to a total $12 + 3\,n - 7$. These are the exterior orientation elements of both photographs, minus the parameters we have eliminated by defining the model coordinate system. By equating the number of equations with number of parameters we obtain the minimum number of points, n_{min}, which we need to measure for solving the orientation problem.

$$4\,n_{\text{min}} = 12 - 7 + 3\,n_{\text{min}} \qquad \Longrightarrow \qquad n_{\text{min}} = 5 \tag{5.29}$$

The collinearity equations which are implicitly referred to in Eq. 5.28 are non-linear. By linearizing the functional form we obtain

$$x, y \approx f^0 + \frac{\vartheta f}{\vartheta X'_C}\Delta X'_C + \frac{\vartheta f}{\vartheta Y'_C}\Delta Y'_C + \cdots + \frac{\vartheta f}{\vartheta Z''_C}\Delta Z''_C \tag{5.30}$$

with f^0 denoting the function with initial estimates for the parameters.

For a point P_i, $i = 1, \cdots, n$ we obtain the following four generic observation equations

$$
\begin{aligned}
r'_{xi} &= \frac{\vartheta f}{\vartheta X'_C}\Delta X'_C + \frac{\vartheta f}{\vartheta Y'_C}\Delta Y'_C + \cdots + \frac{\vartheta f}{\vartheta Z''_C}\Delta Z''_C + f^0 - x'_i \\
r'_{yi} &= \frac{\vartheta f}{\vartheta X'_C}\Delta X'_C + \frac{\vartheta f}{\vartheta Y'_C}\Delta Y'_C + \cdots + \frac{\vartheta f}{\vartheta Z''_C}\Delta Z''_C + f^0 - y'_i \\
r''_{xi} &= \frac{\vartheta f}{\vartheta X'_C}\Delta X'_C + \frac{\vartheta f}{\vartheta Y'_C}\Delta Y'_C + \cdots + \frac{\vartheta f}{\vartheta Z''_C}\Delta Z''_C + f^0 - x''_i \\
r''_{yi} &= \frac{\vartheta f}{\vartheta X'_C}\Delta X'_C + \frac{\vartheta f}{\vartheta Y'_C}\Delta Y'_C + \cdots + \frac{\vartheta f}{\vartheta Z''_C}\Delta Z''_C + f^0 - y''_i
\end{aligned}
\tag{5.31}
$$

As mentioned earlier, the definition of the model coordinate system reduces the number of parameters by seven. Several techniques exist to consider this in the least squares approach.

1. The simplest approach is to eliminate the parameters from the parameter list. We will use this approach for discussing the dependent and independent relative orientation.

2. The knowledge about the 7 parameters can be introduced in the mathematical model as seven independent pseudo observations (e.g. $\Delta X_C = 0$), or as condition equations which are added to the normal equations. This second technique is more flexible and it is particularly suited for computer implementation.

5.5.2 Dependent Relative Orientation

The definition of the model coordinate system in the case of a dependent relative orientation is depicted in Fig. 5.10. The position and the orientation is identical to one of the two photo-coordinate systems, say the primed system. This step amounts to introducing the exterior orientation of the photo-coordinate system as known. That is, we can eliminate it from the parameter list. Next, we define the scale of the model coordinate system. This is accomplished by defining the distance between the two perspective centers (base), or more precisely, by defining the X-component.

With this definition of the model coordinate system we are left with the following functional model

$$
x, y = f(\underbrace{ym''_c, zm''_c, \omega'', \phi'', \kappa''}_{\text{ext. or}''}, \underbrace{xm_1, ym_1, zm_1}_{\text{model pt 1}}, \cdots, \underbrace{xm_n, ym_n, zm_n}_{\text{model pt n}}) \tag{5.32}
$$

With 5 points we obtain 20 observation equations. On the other hand, there are 5 exterior orientation parameters and 5×3 model coordinates. Usually more than 5 points are measured. The redundancy is $r = n - 5$. The typical case of relative orientation

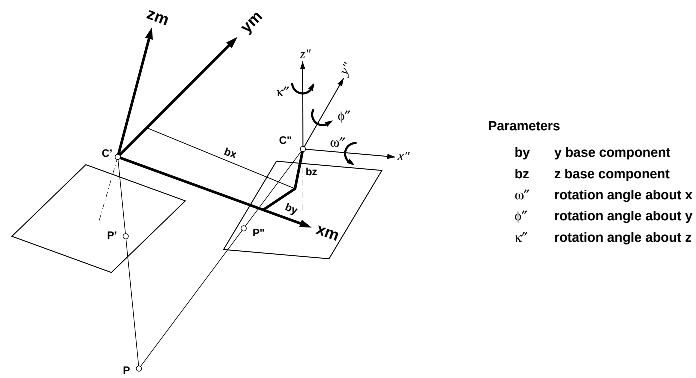

Figure 5.10: Definition of the model coordinate system and orientation parameters in the dependent relative orientation.

on a stereoplotter with the 6 von Gruber points leads only to a redundancy of one. It is highly recommended to measure more, say 12 points, in which case we find $r = 7$.

With a non linear mathematical model we need be concerned with suitable approximations to ensure that the iterative least squares solution converges. In the case of the dependent relative orientation we have

$$f^0 = f(yc_c^0, zm_c^0, \omega^0, \phi^0, \kappa^0, xm_1^0, ym_1^0, zm_1^0, \cdots, xm_n^0, ym_n^0, zm_n^0) \qquad (5.33)$$

The initial estimates for the five exterior orientation parameters are set to zero for aerial applications, because the orientation angles are smaller than five degrees, and $xm_c >> ym_c,\ xm_c >> zm_c \implies ym_c^0 = zm_c^0 = 0$. Initial positions for the model points can be estimated from the corresponding measured photo-coordinates. If the scale of the model coordinate system approximates the scale of the photo-coordinate system, we estimate initial model points by

$$\begin{array}{rcl} xm_i^0 & \approx & x_i' \\ ym_i^0 & \approx & y_i' \\ zm_i^0 & \approx & z_i' \end{array} \qquad (5.34)$$

The dependent relative orientation leaves one of the photographs unchanged; the other one is oriented with respect to the unchanged system. This is of advantage for the conjunction of successive photographs in a strip. In this fashion, all photographs of a strip can be joined into the coordinate system of the first photograph.

5.5.3 Independent Relative Orientation

Fig. 5.11 illustrates the definition of the model coordinate system in the independent relative orientation.

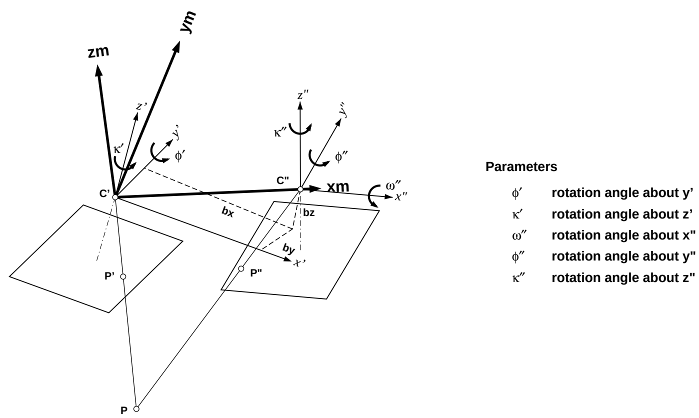

Figure 5.11: Definition of the model coordinate system and orientation parameters in the independent relative orientation.

The origin is identical to one of the photo-coordinate systems, e.g. in Fig. 5.11 it is the primed system. The orientation is chosen such that the positive xm-axis passes through the perspective center of the other photo-coordinate system. This requires determining two rotation angles in the primed photo-coordinate system. Moreover, it eliminates the base components *by, bz*. The rotation about the x-axis (ω) is set to zero. This means that the ym-axis is in the $x-y$ plane of the photo-coordinate system. The scale is chosen by defining $xm''_c = bx$.

With this definition of the model coordinate system we have eliminated the position of both perspective centers and one rotation angle. The following functional model applies

$$x, y = f(\underbrace{\phi', \kappa'}_{\text{ext.or.}'}, \underbrace{\omega'', \phi'', \kappa''}_{\text{ext.or.}''}, \underbrace{xm_1, ym_1, zm_1}_{\text{model pt 1}}, \cdots, \underbrace{xm_n, ym_n, zm_n}_{\text{model pt n}}) \tag{5.35}$$

The number of equations, number of parameters and the redundancy are the same as in the dependent relative orientation. Also, the same considerations regarding initial estimates of parameters apply.

Note that the exterior orientation parameters of both types of relative orientation are related. For example, the rotation angles ϕ', κ' can be computed from the spatial direction of the base in the dependent relative orientation.

$$\phi' = \arctan(\frac{zm''_c}{bx}) \tag{5.36}$$

$$\kappa' = \arctan(\frac{ym''_c}{(bx^2 + zm_c^2)^{1/2}}) \tag{5.37}$$

5.5.4 Direct Orientation

In the direct orientation, the model coordinate system becomes identical with the ground system, for example, a State Plane coordinate system (see Fig. 5.12). Since such systems are already defined, we cannot introduce a priori information about exterior orientation parameters like in both cases of relative orientation. Instead we use information about some of the object points. Points with known coordinates are called *control points*. A point with all three coordinates known is called *full control point*. If only X and Y is known then we have a *planimetric control point*. Obviously, with an *elevation control point* we know only the Z coordinate.

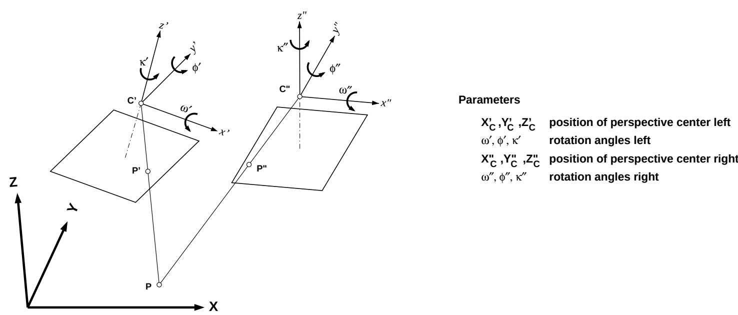

Figure 5.12: Direct orientation of a stereopair with respect to a ground control coordinate system.

The required information about 7 independent coordinates may come from different arrangements of control points. For example, 2 full control points and an elevation, or two planimetric control points and three elevations, will render the necessary information. The functional model describing the latter case is given below:

$$x, y = f(\underbrace{X'_C, Y'_C, Z'_C, \omega', \phi', \kappa'}_{\text{ext. or}'}, \underbrace{X''_C, Y''_C, Z''_C, \omega'', \phi'', \kappa''}_{\text{ext. or}''}, \underbrace{Z_1, Z_2, X_3, Y_3, X_4, Y_4, X_5, Y_5}_{\text{unknown coord. of ctr. pts}}) \tag{5.38}$$

The Z-coordinates of the planimetric control points 1 and 2 are not known and thus remain in the parameter list. Likewise, $X - Y$-coordinates of elevation control points $3, 4, 5$ are parameters to be determined. Let us check the number of observation equations for this particular case. Since we measure the five partial control points on both photographs we obtain 20 observation equations. The number of parameters amounts to 12 exterior orientation elements and 8 coordinates. So we have just enough equations to solve the problem. For every additional point 4 more equations and 3 parameters are added. Thus, the redundancy increases linearly with the number of points measured. Additional control points increase the redundancy more, e.g. full control points by 4, an elevation by 2.

Like in the case of relative orientation, the mathematical model of the direct orientation is also based on the collinearity equations. Since it is non-linear in the parameters we need good approximations to assure convergence. The estimation of initial values for the exterior orientation parameters may be accomplished in different ways. To estimate X_C^0, Y_C^0 for example, one could perform a 2-D transformation of the photo coordinates to planimetric control points. This would also result in a good estimation of κ^0 and of the photo scale which in turn can be used to estimate $Z_C^0 =$ scale c. For aerial applications we set $\omega^0 = \phi^0 = 0$. With these initial values of the exterior orientation one can compute approximations X_i^0, Y_i^0 of object points where $Z_i^0 = h_{\text{aver}}$.

Note that the minimum number of points to be measured in the relative orientation is 5. With the direct orientation, we need only three points assuming that two are full control points. For orienting stereopairs with respect to a ground system, there is no need to first perform a relative orientation followed by an absolute orientation. This traditional approach stems from analog instruments where it is not possible to perform a direct orientation by mechanical means.

5.5.5 Absolute Orientation

With absolute orientation we refer to the process of orienting a stereomodel to the ground control system. Fig. 5.13 illustrates the concept. This is actually a very straightforward task which we discussed earlier under 7-parameter transformation. Note that the 7-parameter transformation establishes the relationship between two 3-D Cartesian coordinate systems. The model coordinate system is cartesian, but the ground control system is usually not cartesian because the elevations refer to a separate datum. In that case, the ground control system must first be transformed into an orthogonal system.

The transformation can only be solved if a priori information about some of the parameters is introduced. This is most likely done by control points. The same considerations apply as just discussed for the direct orientation.

From Fig. 5.13 we read the following vector equation which relates the model to the ground control coordinate system:

$$\mathbf{p} = s\mathbf{R}\mathbf{pm} - \mathbf{t} \tag{5.39}$$

where $\mathbf{pm} = [xm, ym, zm]^T$ is the point vector in the model coordinate system, $\mathbf{p} = [X, Y, Z]^T$ the vector in the ground control system pointing to the object point P and $\mathbf{t} = [X_t, Y_t, Z_t]^T$ the translation vector between the origins of the 2 coordinate systems. The rotation matrix $\mathbf{R}$ rotates vector $\mathbf{pm}$ into the ground control system and s, the scale factor, scales it accordingly. The 7 parameters to be determined comprise 3 rotation angles of the orthogonal rotation matrix **R**, 3 translation parameters and one scale factor.

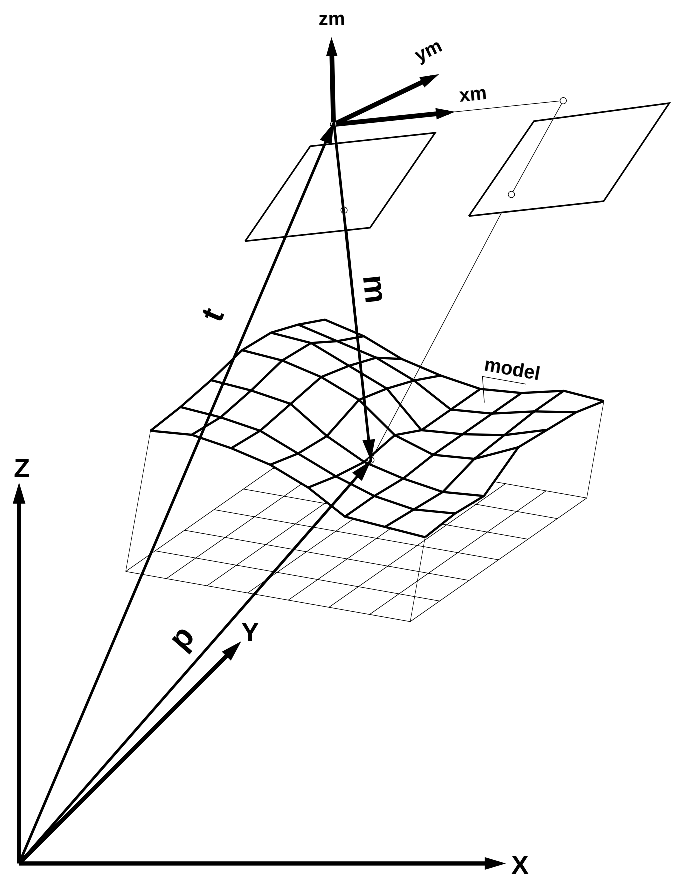

Figure 5.13: Absolute orientation entails the computation of the transformation parameters between model and ground coordinate system.

The following functional model applies:

$$x, y, z = f(\underbrace{X_t, Y_t, Z_t}_{\text{translation}}, \underbrace{\omega, \phi, \kappa}_{\text{orientation}}, \underbrace{s}_{\text{scale}}) \tag{5.40}$$

In order to solve for the 7 parameters at least seven equations must be available. For example, 2 full control points and one elevation control point would render a solution. If more equations (that is, more control points) are available then the problem of determining the parameters can be cast as a least-squares adjustment. Here, the idea is to minimize the discrepancies between the transformed and the available control points. An observation equation for control point P_i in vector form can be written as

$$\mathbf{r_i} = s\mathbf{R}\mathbf{p}\mathbf{m_i} - \mathbf{t} - \mathbf{p_i} \tag{5.41}$$

with $\mathbf{r}$ the residual vector $[r_x, r_y, r_z]^T$. Obviously, the model is not linear in the parameters. As usual, linearized observation equations are obtained by taking the partial derivatives with respect to the parameters. The linearized component equations are

The approximations may be obtained by first performing a 2-D transformation with x, y-coordinates only.